Two regularizations – two different models of Nambu–Jona-Lasinio

R.G. Jafarov[*]and V.E. Rochev [†]

Abstract

Two variants of the Nambu–Jona-Lasinio model – the model with 4-dimensional cutoff and the model with dimensionally-analytical regularization – are systematically compared. It is shown that they are, in essence, two different models of light-quark interaction. In the mean-field approximation the distinction becomes apparent in a behavior of scalar amplitude near the threshold. For 4-dimensional cutoff the pole term can be extracted, which corresponds to sigma-meson. For dimensionally-analytical regularization the singularity of the scalar amplitude is not pole, and this singularity is quite disappeared at some value of the regularization parameter.

Still more essential distinction of these models exists in the next-to-leading order of mean-field expansion. The calculations of meson contributions in the quark chiral condensate and in the dynamical quark mass demonstrate, that these contributions though their relatively smallness can destabilize the Nambu–Jona-Lasinio model with 4-dimensional cutoff. On the contrary, the Nambu–Jona-Lasinio model with dimensionally-analytical regularization is stabilized with the next-to-leading order, i.e. the value of the regularization parameter shifts to the stability region, where these contributions decrease.

---

[*]Baku State University, Baku, Azerbaijan

[†]Institute for High Energy Physics, Protvino, Russia (e-mail: rochev@mx.ihep.su)

Introduction

The Namby–Jona-Lasinio (NJL) model [1] with the quark content [2] is one of the most successful effective models of quantum chromodynamics of light hadrons in the non-perturbative region (see, for example, reviews [3] and [4] and references therein).

Since the foundation of the NJL model is a non-renormalizable interaction, the quite essential point of the model is a regularization. It already advances in the literature an opinion, that the NJL model for different regularization can lead to different physical results. But as concerning to most common regularizations (such, for example, the 4-dimensional cutoff in comparison with the Fock–Schwinger "proper-time" regularization or the Pauli–Villars regularization) this statement is not mean some principal distinctions of main effects in the leading approximation of the model. In the next-to-leading order, which includes the meson contributions in chiral condensate and corrections to the quark propagator, these distinctions become apparent more clearly (see, for example, [5] - [7]), but do not change essentially the physical content of the model in the case too.

Nevertheless, a regularization of the NJL model exists in which the physical effects differ from the effects of the classical variant of the model with 4-dimensional cutoff as early as on the level of two-particle amplitudes. It is a dimensional regularization considered as a variant of the analytical regularization. Note, the traditional treatment of the dimensional regularization as a transition to D-dimensional space strikes in the application to the NJL model the essential obstacle: the regularization parameter, i.e. a deviation in physical dimension of space, is included in formulae for physical quantities. This circumstance makes an interpretation of results to be very awkward. In the alternative treatment of dimensional regularization as a variant of analytical regularization all calculations are made in four-dimensional Euclidean space, and the regularization parameter is treated as a power of a weight function, which regularizes divergent integrals[1]. Such treatment of dimensional regularization was consistently developed for the NJL model in mean-field approximation by Krewald and Nakayama [8]. In work [9] in the framework of this regularization the meson contributions in chiral condensate were calculated. It should be stressed that in such treatment of dimensional regularization the regularization parameter is not a deviation in the physical dimension of space. A possible treatment of this parameter (see [9]) is a measure of some effective gluon influence on four-fermion quark self-action of NJL model.

In this work we make a systematical comparison of the dimensionally-analytically regularized NJL model with the classical variant of NJL model with 4-dimensional cutoff[2].

In Sections 1 and 2 the leading-approximation results for the chiral condensate and two-particle amplitudes are given. At this point the distinction becomes apparent in scalar amplitude. For 4-dimensional cutoff and for other similar regularizations the scalar amplitude holds the singularity of pole type, which is usually interpreted as a sigma-meson with mass $2m$, where m is a quark mass. But for the dimensionally-analytical regularization the singularity of scalar amplitude is not a pole (moreover, for some value of the regularization parameter this singularity quite disappears), and the particle interpretation seems to be inconsistent. Note, that the interpretation of singularity of scalar amplitude in the NJL model as a particle meets with the well-known difficulties in a comparison with the physical spectrum of scalar-meson resonances (see, for example, [10]).

---

[1]We shall refer to this regularization as dimensionally-analytical regularization. In applications to renormalizable models this variant of dimensional regularization leads to same results as the usual treatment.

[2]We consider here the SU(2)-NJL model, i.e. the NJL model with chiral $SU_V(2) \times SU_A(2)$-symmetry.

A main distinction becomes apparent in the next-to-leading order of mean-field expansion, therefore the main results are calculations made in this order in Section 3. Apart from the corrections to chiral condensate we calculate also the corrections to quark mass for both regularizations. At that, in contrast to work [9] in which for the scalar amplitude were used a pole approximation, we use here more exact leading-singularity approximation. For both regularizations the leading contribution in first-order chiral condensate is the contribution of pseudoscalar (pion) amplitude. But for the dimensionally-analytical regularization this contribution has the same sign as the leading-order condensate, and for 4-dimensional cutoff it has the opposite sign. This distinction is a determining factor for the problem of stability of the model with respect to quantum fluctuations caused by the meson amplitudes. In Section 4 a fixation of the model parameters are made with the taking into account the meson corrections. It is shown, that the coincidence of sign of the meson contributions with the sign of leading contribution in the model with dimensionally-analytical regularization ensures the stability of the model with respect to quantum fluctuations. Contrary, for the model with 4-dimensional cutoff these corrections have the opposite sign and can lead to a destabilization. This destabilization means that an existence of the set of model parameters critically depends on the value of chiral condensate c: at $|c| \leq 230$ MeV a system of equations for the model parameters has no solution. Therefore the SU(2)-NJL model with 4-dimensional cutoff seems to be in a dangerous zone of the non-stability with respect to quantum fluctuations. A simple estimate demonstrates that for SU(3)-model a situation can merely take a turn for the worse. This result in a sense has something in common with the statement of work [11] in which an applicability of the NJL model to the description of phenomenon of dynamical chiral symmetry breaking (DCSB) was called in question. (See also a discussion of this problem in works [7, 12].) Therefore we can maintain that the NJL model with dimensionally-analytical regularization and the NJL model with 4-dimensional cutoff are, in essence, two different models of the light-quark interaction in non-perturbative region.

1 Leading order and chiral condensate

We consider the NJL model with Lagrangian

$$\mathcal{L} = \bar{\psi} i \hat{\partial} \psi + \frac{g}{2} \Big[(\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma_5 \tau^a \psi)^2 \Big]. \tag{1}$$

Here $\psi \equiv \psi_j^{\alpha,c}$, where $\alpha = 1, 2, 3, 4$ is Dirac spinor index, $c = 1, \ldots, n_c$ is colour index, $j = 1, 2$ is isotopic (flavour) index, τ^a are generators of $SU(2)$-group (Pauli matrices), $a = 1, 2, 3$. This model possesses the chiral symmetry of $SU_V(2) \times SU_A(2)$-group.

The mean-field expansion in bilocal-source formalism [13] for this model is constructed with the scheme which is given in work [9].

In the leading approximation the unique connected Green function is quark propagator

$$S_{cd,jk}^{(0)} = \delta_{cd} \delta_{jk} (m - \hat{p})^{-1}, \tag{2}$$

where dynamical quark mass m is a solution of gap equation

$$1 = -8 i g n_c \int \frac{d\tilde{q}}{m^2 - q^2}. \tag{3}$$

(Here and below we include a phase factor in an integration over momentum space: $d\tilde{q} \equiv d^4q/(2\pi)^4$.)

The basic order parameter, which defines a degree of DCSB, is a quantity

$$\chi = <0|\bar{\psi}\psi|0> = i\,\mathrm{tr}\,S(x)|_{x\to 0},$$

where the trace is taken over all discrete indices. It is easy to see, that from equations (2) and (3) it follows

$$\chi^{(0)} = i\,\mathrm{tr}\,S^{(0)}(x)|_{x\to 0} = -\frac{m}{g}. \tag{4}$$

It is a regularization-independent formula.

Quark chiral condensate c is defined for each flavour separately. In the chiral limit it is

$$c = \left(\frac{\chi}{2}\right)^{1/3}. \tag{5}$$

The integral in equation (3) is divergent, and it should be considered as a regularization. In Euclidean momentum space equation(3) has the form

$$1 = \frac{gn_c}{2\pi^2}\int \frac{q_e^2 dq_e^2}{m^2 + q_e^2}.$$

Let introduce in the integrand a weight function $w(q_e^2)$, which will define a choice of regularization. For 4-dimensional cutoff the weight function is chosen in the form

$$w_\Lambda(q_e^2) = \theta(\Lambda^2 - q_e^2), \tag{6}$$

and equation (3) is

$$1 = \kappa_\Lambda\left(1 - \frac{m^2}{\Lambda^2}\log(1 + \frac{\Lambda^2}{m^2})\right), \tag{7}$$

where $\kappa_\Lambda = gn_c\Lambda^2/2\pi^2$. This equation exactly correspond to the classical result of work [1].

For dimensionally-analytical regularization the weight function is chosen as

$$w_\xi(q_e^2) = \frac{1}{\Gamma(1-\xi)}\left(\frac{M^2}{q_e^2}\right)^{1+\xi}. \tag{8}$$

Integral over dq_e^2 from 0 to ∞ converge at $0 < \xi < 1$, and equation (6) for dimensionally-analytical regularization has the form:

$$1 = \kappa\Gamma(\xi)\left(\frac{M^2}{m^2}\right)^{1+\xi}, \tag{9}$$

where $\kappa = gn_c m^2/2\pi^2$. Factor $\frac{1}{\Gamma(1-\xi)}$ in weight function (8) are introduced for the correspondence of integration results to that of the usual prescription of the dimensional regularization as a formal transition in D-dimensional space. Note, that parameter ξ differs from the commonly used parameter ε, which is defined by relation $D = 4 - 2\varepsilon$. As easy to see, these parameters are connected by relation $\varepsilon = 1 + \xi$. Introduction of this new notation prevents unnecessary associations with the usual treatment of dimensional regularization. Furthermore, in terms of the parameter ξ all subsequent formulae of the NJL model acquire most simple form.

2   Two-particle amplitude and model parameters in leading approximation

First-step two-particle amplitude A (a connected part of amputated two-particle function) possesses the following colour and flavour structure:

$$A^{cd,jk}_{c'd',j'k'} = \delta^{cd}\delta^{c'd'}\left[\delta_{jk}\delta_{j'k'}A_\sigma + \tau^a_{jk}\tau^a_{j'k'}A_\pi\right].\tag{10}$$

Here A_σ is the scalar amplitude, and A_π is the pseudoscalar amplitude. In momentum space these amplitudes of the NJL model depend on a momentum p only, where p is a sum of quark and antiquark momenta. The amplitudes have the form [9]:

$$A_\sigma(p) = -\frac{ig}{1 - L_S(p)},\tag{11}$$

where $L_S(p) = ig\int d\tilde{q}\ \mathrm{tr}\ S^{(0)}(p+q)S^{(0)}(q)$ is the scalar quark loop, and

$$A_\pi(p) = \frac{ig}{1 + L_P(p)},\tag{12}$$

where $L_P(p) = ig\int d\tilde{q}\ \mathrm{tr}\ S^{(0)}(p+q)\gamma_5 S^{(0)}(q)\gamma_5$ is the pseudoscalar quark loop.
  Using identities

$$\frac{m^2 + q^2 + (pq)}{(m^2 - (p+q)^2)(m^2 - q^2)} = \frac{1}{2}\left(-\frac{1}{m^2 - q^2} - \frac{1}{m^2 - (p+q)^2} + \frac{4m^2 - p^2}{(m^2 - (p+q)^2)(m^2 - q^2)}\right),$$

$$\frac{m^2 - q^2 + (pq)}{(m^2 - (p-q)^2)(m^2 - q^2)} = \frac{1}{2}\left(\frac{1}{m^2 - q^2} + \frac{1}{m^2 - (p-q)^2} + \frac{p^2}{(m^2 - (p-q)^2)(m^2 - q^2)}\right)$$

and gap equation (3), it is easy to obtain for A_σ and A_π the following representations:

$$A_\sigma(p) = \frac{1}{4n_c I_0(p^2)(4m^2 - p^2)},\tag{13}$$

$$A_\pi(p) = \frac{1}{4n_c I_0(p^2)p^2}.\tag{14}$$

Here

$$I_0(p^2) = \int d\tilde{q}\frac{1}{(m^2 - (p+q)^2)(m^2 - q^2)}.\tag{15}$$

Integral I_0 is calculated as above. Transforming to Euclidean metric, introducing a standard Feynman parameterization, and changing an integration variable (which is possible due to translational invariance of the procedure, see [8]), we can perform the angular integration. According to the our rules, then we introduce into the integrand a weight function (6) (for 4-dimensional cutoff), or (8) (for dimensionally-analytical regularization), and calculate the integral over dq_e^2. For dimensionally-analytical regularization (DAR) we again obtain the result, which corresponds to the result of integration with the formal transition to D-dimensional space:

$$I_0^{DAR}(p^2) = \frac{i\Gamma(1+\xi)}{(4\pi)^2}\int_0^1 du\left(\frac{M^2}{m^2 - u(1-u)p^2}\right)^{1+\xi}.$$

The integral over dq_e^2 converges at $-1 < \xi < 1$. Taking into account gap equation (9) we obtain:

$$I_0^{DAR}(p^2) = \frac{i}{(4\pi)^2}\frac{\xi}{\kappa}\int_0^1 du\left(1 - u(1-u)\frac{p^2}{m^2}\right)^{-1-\xi} = \frac{i}{(4\pi)^2}\frac{\xi}{\kappa}F(1+\xi, 1; 3/2; \frac{p^2}{4m^2}), \quad (16)$$

where $F(a, b; c; z)$ is Gauss hypergeometric function.

For 4-dimensional cutoff (*FDC*) we correspondingly obtain:

$$I_0^{FDC}(p^2) = \frac{i}{(4\pi)^2}\int_0^1 du\left[\log\left(1 + \frac{\Lambda^2}{m^2 - u(1-u)p^2}\right) - \frac{\Lambda^2}{\Lambda^2 + m^2 - u(1-u)p^2}\right]. \quad (17)$$

Formulae for the condensate and the two-particle amplitudes allow to fix values of the model parameters in the leading approximation of mean-field expansion. For this purpose we use regularization-independent formulae (4), (5) and a formula for pion decay constant in the NJL model (see [3]):

$$f_\pi^2 = -4in_c m^2 I_0(0). \quad (18)$$

For dimensionally-analytical regularization we obtain from (16):

$$I_0^{DAR}(0) = \frac{i}{(4\pi)^2}\frac{\xi}{\kappa}, \quad (19)$$

and for 4-dimensional cutoff (see (17)):

$$I_0^{FDC}(0) = \frac{i}{(4\pi)^2}\left[\log\frac{\Lambda^2 + m^2}{m^2} - \frac{\Lambda^2}{\Lambda^2 + m^2}\right]. \quad (20)$$

Correspondingly we obtain for dimensionally-analytical regularization very simple formula:

$$(f_\pi^2)^{DAR} = \frac{\xi}{2g}. \quad (21)$$

For 4-dimensional cutoff the analogous formula is

$$(f_\pi^2)^{FDC} = \frac{3m^2}{4\pi^2}\left[\log(1 + \frac{\Lambda^2}{m^2}) - \frac{\Lambda^2}{m^2 + \Lambda^2}\right]. \quad (22)$$

These formulae together with formulae (4)-(5) for condensate and gap equation (equation (9) for dimensionally-analytical regularization and equation (7) for 4-dimensional cutoff) allow to define the values of principal model parameters.

For pion decay constant we choose the value $f_\pi = 93$ MeV. Chiral quark condensate c is not directly measured value, and we shall determine sets of parameters for some typical values of this quantity. For dimensionally-analytical regularization it is necessary also to fix a value of M ("subtraction point"). In work [9] we have used for this purpose a value of decay width $\pi^0 \to 2\gamma$. Analysis of results of this work demonstrates, that for very large range of condensate values the value of M is practically permanent and coincides with the value of dynamical quark mass: $M \approx m$. Since here we shall take $M = m$. Such fixation of M equalizes the parameter number of dimensionally-analytical regularization with that of other regularizations. Gap equation (9) with such fixation of M takes on a very simple form

$$1 = \kappa\Gamma(\xi). \quad (23)$$

The results of parameter fixing in the leading approximation (at $n_c = 3$) are given in Table 1 (dimensionally-analytical regularization) and in Table 2 (4-dimensional cutoff).

c (MeV)	m (MeV)	ξ	$\kappa = 3gm^2/2\pi^2$
-210	357	0.333	0.373
-220	356	0.289	0.322
-230	354	0.252	0.277
-240	353	0.221	0.242
-250	352	0.195	0.212

Table 1. The model parameters in leading order (dimensionally-analytical regularization): chiral condensate c, quark mass m, regularization parameter ξ and dimensionless coupling κ.

c (MeV)	m (MeV)	Λ (MeV)	$\kappa_\Lambda = 3g\Lambda^2/2\pi^2$
-210	423	733	1.86
-220	323	791	1.448
-230	276	873	1.315
-240	253	947	1.240
-250	236	1029	1.187

Table 2. The model parameters in leading order (4-dimensional cutoff): chiral condensate c, quark mass m, regularization parameter Λ and dimensionless coupling κ_Λ.

As it is seen from these Tables the value of the main parameter – quark mass m – in the model with 4-dimensional cutoff is much more sensitive to the value of chiral condensate in comparison with that of the model with dimensionally-analytical regularization. At the same time it is necessary to point, that there are no some principal distinctions of these variants of the NJL model at the level of leading approximation for quark propagator and two-particle amplitudes with the exception of a behavior of scalar amplitude A_σ in threshold region. Consider this point in more details.

Pseudoscalar amplitude A_π naturally associates with the pion, which in the chiral limit is a massless Goldstone particle. In both regularizations under consideration we can define a pion propagator as a pole term of A_π, which corresponds to the leading singularity of pseudoscalar amplitude:

$$A_\pi^{pole}(p) = \frac{1}{4n_c I_0(0)p^2},\tag{24}$$

where $I_0(0)$ is defined by equation (19) for dimensionally-analytical regularization and by (20) for 4-dimensional cutoff.

For the scalar amplitude the situation is different. In both regularizations function $I_0(p^2)$ possesses a cut which originates in the point $p^2 = 4m^2$. Nevertheless, for 4-dimensional cutoff it is possible to define a scalar sigma-meson propagator as

$$A_\sigma^{pole}(p) = \frac{1}{4n_c I_0(4m^2)(4m^2 - p^2)},\tag{25}$$

6

since

$$I_0^{FDC}(4m^2) = \frac{i}{(4\pi)^2}\Big[\log\frac{\Lambda^2 + m^2}{m^2} + \frac{\Lambda}{m}\arctan\frac{m}{\Lambda}\Big]$$

is a finite quantity. But for dimensionally-analytical regularization this quantity is finite only at $\xi < -1/2$:

$$I_0^{DAR}(4m^2)|_{\xi<-1/2} = -\frac{i}{8gn_cm^2}\frac{\xi}{1 + 2\xi}.$$

For an interpretation of the sigma-meson as a particle in the NJL model with dimensionally-analytical regularization we can do the following trick: since in the region $-1 < \xi < -1/2$ integral I_0 converges we use the above value in the point $p^2 = 4m^2$ as a foundation for an analytical continuation of the pole part of the amplitude on parameter ξ to the physical region $0 < \xi < 1$. Then the sigma-meson propagator for dimensionally-analytical regularization would be

$$(A_\sigma^{pole}(p))^{DAR} = \frac{2igm^2(1 + 2\xi)}{(4m^2 - p^2)\xi}. \tag{26}$$

This expression was used for a calculation of the sigma-meson contribution in chiral condensate in work [9]. Surely, such procedure of definition of sigma-meson propagator seems to be a somewhat artificial. A more consistent procedure is a separation of a leading singular part of amplitude in the region of physical values of regularization parameter ξ.

For the pseudoscalar amplitude the separation of leading singularity near the point $p^2 = 0$ leads to same result (24), i.e. the pion in dimensionally-analytical regularization possesses all properties of usual observable particle. For the scalar amplitude it is not so. At $p^2 \to 4m^2$ in region $0 < \xi < 1$:

$$I_0^{DAR} \cong \frac{i\sqrt{\pi}\Gamma(\xi + 1/2)}{16gn_cm^2\Gamma(\xi)} \cdot \left(\frac{4m^2}{4m^2 - p^2}\right)^{\xi+1/2},$$

and, correspondingly, the leading singularity (LS), i.e. a leading term in an expansion on powers of $4m^2 - p^2$ is the expression

$$(A_\sigma^{LS})^{DAR} \cong -\frac{ig\Gamma(\xi)}{\sqrt{\pi}\Gamma(\xi + 1/2)} \cdot \left(\frac{4m^2}{4m^2 - p^2}\right)^{1/2-\xi}. \tag{27}$$

Thus, the leading singularity of scalar amplitude in the model with dimensionally-analytical regularization is of the principally different type in comparison with the cutoff model. Instead of the pole term, which can be naturally interpret as sigma-particle propagator, we obtain in dimensionally-analytical regularization the power behavior which depends on the regularization parameter ξ. Moreover, due to formula

$$F(3/2, 1; 3/2; \frac{p^2}{4m^2}) = \frac{4m^2}{4m^2 - p^2}$$

at $\xi = 1/2$ a cancellation of contributions into two-particle amplitude takes place (see also [14]). At this parameter value ($\xi = 1/2$) we obtain for the amplitudes extremely simple expressions:

$$A_\pi = ig - \frac{4igm^2}{p^2}, \quad A_\sigma = -ig, \tag{28}$$

7

i.e. at $\xi = 1/2$ the scalar amplitude has no singularity – sigma-meson disappears! We emphasize, that result (28) is an exact consequence of formulae (13), (14) and (16) without any approximation type of above leading-singularity approximation.

Thus, we come to the conclusion, that for dimensionally-analytical regularization at physical values of parameters the scalar amplitude A_σ does not possesses a pole term, which can be interpret as a physical scalar meson.

3   Meson contributions in chiral condensate and in quark propagator

First-order equations of the mean-field expansion (see [9]) define corrections to quark propagator. First-order mass operator $\Sigma^{(1)} = S_0^{-1} \star S^{(1)} \star S_0^{-1}$, where $S^{(1)}$ is a first-order correction to quark propagator, is defined in x-space by equation (see **??**)

$$\Sigma^{(1)}(x) = S^{(0)}(x)A_\sigma(x) + 3S^{(0)}(-x)A_\pi(x) + ig\delta(x)\operatorname{tr} S^{(1)}(0). \tag{29}$$

Introducing dimensionless first-order mass functions $a^{(1)}$ and $b^{(1)}$:

$$\Sigma^{(1)} \equiv a^{(1)}\hat{p} - b^{(1)}m, \tag{30}$$

and also defining the first-order condensate

$$\chi^{(1)} = i\operatorname{tr}S^{(1)}(0) \tag{31}$$

and a ratio of the first-order condensate to the leading-order condensate

$$r \equiv \frac{\chi^{(1)}}{\chi^{(0)}},$$

we obtain from (29) the expressions for $a^{(1)}$ and $b^{(1)}$ in momentum space:

$$p^2 a^{(1)}(p^2) = \int d\tilde{q}\frac{p^2 - (pq)}{m^2 - (p-q)^2}[A_\sigma(q) - 3A_\pi(q)], \tag{32}$$

$$b^{(1)}(p^2) = r - \int \frac{d\tilde{q}}{m^2 - (p-q)^2}[A_\sigma(q) + 3A_\pi(q)]. \tag{33}$$

It follows from equations (32) and (33), that the corrections to quark propagator consist of two parts: pion correction (due to pseudoscalar amplitude A_π) and contribution due to scalar amplitude A_σ: $a^{(1)} = a_\pi^{(1)} + a_\sigma^{(1)}$; $b^{(1)} = b_\pi^{(1)} + b_\sigma^{(1)}$.

For the ratio of the first-order condensate (31) to the leading-order condensate (4) we obtain the formula

$$r = -\frac{g\chi^{(1)}}{m} = -8ign_c \int d\tilde{p}\frac{2p^2 a_1 - (m^2 + p^2)b_1}{(m^2 - p^2)^2}. \tag{34}$$

Corrections to quark mass can be found after the calculation the condensate corrections. Inverse quark propagator is

$$S^{-1} = m - \hat{p} - \Sigma^{(1)} = b(p^2) - a(p^2)\hat{p} = (1 + b^{(1)})m - (1 + a^{(1)})\hat{p}.$$

8

Suppose the propagator has a pole in point $p^2 = m_r^2$, which corresponds to a particle with mass m_r. Then

$$b(m_r^2) = m_r a(m_r^2).$$

Since $a^{(1)}$ and $b^{(1)}$ are small additions, we can to expand $a^{(1)}(m_r^2)$ and $b^{(1)}(m_r^2)$ near the point m and to obtain the formula for the quark-mass correction $\delta m \equiv m_r - m$:

$$\frac{\delta m}{m} \cong b^{(1)}(m^2) - a^{(1)}(m^2). \tag{35}$$

3.1 Pion contribution

The pion contribution to quark propagator is defined by formulae

$$p^2 a_\pi^{(1)}(p^2) = -3 \int d\tilde{q} \frac{p^2 - (pq)}{m^2 - (p-q)^2} A_\pi(q), \tag{36}$$

$$b_\pi^{(1)}(p^2) = r_\pi - 3 \int \frac{d\tilde{q}}{m^2 - (p-q)^2} A_\pi(q). \tag{37}$$

For the calculation we shall use the pole approximation (24). The calculation reduces to calculations of integrals

$$I_0(p^2; m^2, \mu^2) = \int \frac{d\tilde{q}}{(m^2 - (p-q)^2)(\mu^2 - q^2)}, \tag{38}$$

$$I_\nu(p^2; m^2, \mu^2) = \int \frac{q_\nu d\tilde{q}}{(m^2 - (p-q)^2)(\mu^2 - q^2)} \tag{39}$$

at $\mu^2 \to 0$.

These integrals are calculated with above rules (see Sections 2 and 3). The pion contribution to first-order condensate is calculated by formula (34). Integral can be calculated in a closed form, and the result is the very simple expression:

$$(r_\pi)^{DAR} = \frac{3}{8 n_c \xi}. \tag{40}$$

(See also [9], where this result have been obtained by a slightly different method.)

To calculate the pion contribution for 4-dimensional cutoff we use equations (24) and (20). Further the pion contribution in condensate is calculated with formula (34). For 4-dimensional cutoff the result for r_π is not described by a simple formula, as for dimensionally-analytical regularization. Nevertheless, the computation has no any troubles. Note, that whereas in dimensionally-analytical regularization r_π is a function of regularization parameter ξ, in 4-dimensional cutoff this quantity is a function of ratio $x \equiv \Lambda^2/m^2$:

$$(r_\pi)^{FDC} = r_\pi(\Lambda^2/m^2).$$

As examples we give values of $(r_\pi)^{FDC}$ for two characteristic values of this ratio[3]. At $x = 3$ (this value corresponds to value $c^{(0)} = -210$MeV of the leading-order condensate) the computation gives $(r_\pi)^{FDC} = -0.272$. At $x = 19$ (this value corresponds to value $c^{(0)} = -250$MeV of the leading-order condensate) the computation gives $(r_\pi)^{FDC} = -0.183$.

---

[3]All values given here correspond to physical value of colours $n_c = 3$.

Let turn to the pion contribution in quark mass. To calculate this contribution we apply equation (35). As a result we obtain (with taking into account gap equation (23) and equation (40)):

$$\left(\frac{\delta m_{(\pi)}}{m}\right)^{DAR} = (r_\pi)^{DAR} - \frac{3}{8n_c\xi} = 0, \tag{41}$$

i.e. for the dimensionally-analytical regularization the pion correction to quark mass equal zero.

For 4-dimensional cutoff the pion correction to quark mass is

$$\left(\frac{\delta m_{(\pi)}}{m}\right)^{FDC} = (r_\pi)^{FDC} + \frac{3}{n_c}h_\pi(\Lambda^2/m^2), \tag{42}$$

where

$$h_\pi(x) = \frac{\log(1+x)}{8[\log(1+x) - \frac{x}{1+x}]}.$$

Signs of $(r_\pi)^{FDC}(x)$ and $h_\pi(x)$ are opposite, and their contributions in δm are mutually cancelled. Moreover, as for the dimensionally-analytical regularization, the pion correction to quark mass equal zero. We obtain this result, unlike to the exact result (41) of dimensionally-analytical regularization, by computations in a framework of given accuracy of inputs. Such coincidence of results in both regularizations suggests an idea that the zero value of the pion correction to quark mass is the regularization-independent fact of NJL model.

3.2 Scalar contribution

Consider a contribution of scalar amplitude A_σ in condensate and quark mass. In correspondence with (33) and (32) we have

$$p^2 a_\sigma^{(1)} = \int d\tilde{q}\frac{p^2 - (pq)}{m^2 - (p-q)^2}A_\sigma(q), \tag{43}$$

$$b_\sigma^{(1)} = r_\sigma - \int \frac{d\tilde{q}}{m^2 - (p-q)^2}A_\sigma(q). \tag{44}$$

To calculate this contribution we use the leading-singularity approximation:

$$A_\sigma^{LS} = \frac{1}{4n_c(4m^2 - p^2)I_0(p^2)|_{p^2\to 4m^2}}.$$

For the dimensionally-analytical regularization this approximation is described by equation (27). From equation (34) we obtain the quantity r_σ. A computation gives us the following values for sigma-contribution: at $\xi = 0.25$ we obtain $(r_\sigma)^{DAR} = -0.033$; at $\xi = 0.4$ we obtain $(r_\sigma)^{DAR} = -0.01$. As one can see, the sigma-contribution is small in comparison of the pion contribution and possesses the opposite sign, i.e. it decrease the common contribution[4].

For the 4-dimensional cutoff the leading-singularity approximation for A_σ coincides with the pole approximation(25). Then r_σ is calculated by equation (34). This quantity, as r_π, for the 4-dimensional cutoff is a function of $x \equiv \Lambda^2/m^2$:

$$(r_\sigma)^{FDC} = r_\sigma(\Lambda^2/m^2).$$

___

[4]Note, that this result is qualitatively the same as result of work [9], in which has been used a pole approximation for A_σ. Thus, all conclusions of work [9] about the part of the meson contributions stand also for the more exact leading-singularity approximation, which is used in present work

At $x = 3$ we obtain $(r_\sigma)^{FDC} = -0.007$. At $x = 19$ we obtain $(r_\sigma)^{FDC} = -0.116$. In contrast to the dimensionally-analytical regularization, the sign of sigma-contribution for the 4-dimensional cutoff is the same as for pion contribution.

A sigma-correction to quark mass for dimensionally-analytical regularization is given by formula

$$\left(\frac{\delta m_{(\sigma)}}{m}\right)^{DAR} = (r_\sigma)^{DAR} - \frac{\cos \pi\xi}{4^{1+\xi} n_c \pi (1/2 - \xi)} \tag{45}$$

and attains: at $\xi = 0.25$: $\delta m_{(\sigma)}^{DAR} = -0.086m$, and at $\xi = 0.4$: $\delta m_{(\sigma)}^{DAR} = -0.056m$. Since a pion correction to quark mass in this regularization equals zero (see above), these values are full corrections to quark mass for dimensionally-analytical regularization.

For the 4-dimensional cutoff the sigma-correction to quark-mass is

$$\left(\frac{\delta m_{(\sigma)}}{m}\right)^{FDC} = (r_\sigma)^{FDC} - \frac{1}{n_c} h_\sigma(\Lambda^2/m^2), \tag{46}$$

where

$$h_\sigma(x) = \frac{4\log(1 + x/4) - \log(1 + x)}{8[\log(1 + x) + \sqrt{x} \arctan\sqrt{\frac{1}{x}}]}.$$

At $x = 3$: $\delta m_{(\sigma)}^{FDC} = -0.022m$; at $x = 19$: $\delta m_{(\sigma)}^{FDC} = -0.158m$.

In conclusion of this Section let consider an issue on an accuracy of above calculations. A principal approximation of above calculations is the leading-singularity approximation. Let consider a part of other terms. To estimate their part for dimensionally-analytical regularization let use the simple expressions of amplitudes at $\xi = 1/2$ (see (28)). Remind these expressions are exact. A calculation with formulae (32)–(34) demonstrates that the contributions of non-pole terms in chiral condensate equal zero. Since the values of parameter ξ are near this point (see below, Table 3), we can maintain, that at $\xi \neq 1/2$ their contributions are also small in comparison with the main pole contribution.

For 4-dimensional cutoff the calculations with exact formulae (13)-(14) for the amplitudes also demonstrate, that the leading-singularity approximation (pole approximation in the case) gives the main contribution in condensate. So, at $x = 3$ the calculation with the exact formulae (13)-(14) gives for the pion contribution $r_\pi = -0.267$, i.e., differs from the result of pole approximation (see Subsection 3.1) less then on 2%. For sigma-contribution the difference is more significant: the calculation with the exact formulae gives $r_\sigma = -0.031$, but since this contribution is much less in comparison with the pion contribution, this difference again practically does not affect to final result.

4   Improved model parameters

The condensate and the quark-mass corrections, which were calculated in preceding Section, allow us to specify parameters of the SU(2)-NJL model. We modify a formula for the condensate as follows:

$$\chi = \chi^{(0)} + \chi^{(1)} = -\frac{m}{g}(1 + r). \tag{47}$$

The formula for f_π (see (18)) stays the same, since corrections to amplitudes generate in the next (second) order of mean-field expansion. The quark mass is the mass m_r:

$$m_r = m + \delta m,$$

where δm is defined by equation (35). Values of the model parameters at $n_c = 3$ for this improved choice are given in Tables 3 and 4.

c (MeV)	m_r (MeV)	ξ	$\kappa = 3gm^2/2\pi^2$
-210	339	0.432	0.486
-220	336	0.385	0.434
-230	333	0.346	0.387
-240	330	0.312	0.334
-250	328	0.284	0.316

Table 3. Model parameters with first-order corrections (dimensionally-analytical regularization): chiral condensate c, quark mass m_r, regularization parameter ξ and dimensionless coupling κ.

c (MeV)	m_r (MeV)	Λ (MeV)	$\kappa_\Lambda = 3g\Lambda^2/2\pi^2$
-240	310	785	1.501
-250	283	819	1.408

Table 4. Model parameters with first-order corrections (4-dimensional cutoff): chiral condensate c, quark mass m_r, regularization parameter Λ and dimensionless coupling κ_Λ.

Table 4 does not contain the parameter values at $c = -210$ MeV, $c = -220$ MeV and $c = -230$ MeV. These values are absent due to following reason: the system of equations (47), (22) and (7), which determines these parameters, has no solution at $f_\pi = 93$ MeV and at $|c| \leq 230$ MeV. There is very important circumstance – for 4-dimensional cutoff the meson contributions can destabilize the NJL model. Though these contributions are relatively small (they do not exceed 25% from the leading contribution), but their opposite sign leads to a non-stability of all the system. The situation is very similar to that of pointed in work [11]. Note, that increasing a number of flavours, i.e. for the U(n_f)-NJL model (n_f is a number of flavours), the situation takes a turn for the worse, because a main pseudoscalar contribution is proportional to n_f. At that for dimensionally-analytical regularization the situation is principally different: due to the coincidence of sign of the meson contributions with the sign of leading contribution in condensate for this regularization a stabilization of the model takes place. It is clearly seen from Table 3 – values of regularization parameter ξ increase in comparison with corresponding leading-order values (see Table 1), i.e. shift to a region of stability of model, where these meson contributions decrease.

Conclusion

The results of present work demonstrate that the NJL model with dimensionally-analytical regularization essentially differs from the NJL model with 4-dimensional cutoff at least in two aspects.

Firstly, there is the different behavior of scalar amplitude in threshold region. For the 4-dimensional cutoff it is possible to separate near the threshold a pole term, which is usually associated with a scalar particle – sigma-meson (note, however, that reasoning doubts in such interpretation have been stated as early as in founder's work [1]). For the dimensionally-analytical regularization the singularity of scalar amplitude is not of pole type at physical values of regularization parameter. This fact, even if does not exclude entirely, makes its interpretation as a physical particle to be awkward.

But much more principal thing is the different behavior of these models with respect to quantum fluctuations caused by meson contributions in chiral condensate. As it follows from results of Sections 3 and 4, the NJL model with dimensionally-analytical regularization is stable with respect to these fluctuations, whereas for the NJL model with 4-dimensional cutoff the meson contributions can lead to destabilization. Surely, a number of physical applications of the NJL model are connected exclusively with the leading order of mean-field expansion (mean-field approximation), for which the possibility of such destabilization can be simply ignored. On the other hand, some physical applications of the NJL model exist, that connected with multi-quark functions (such as pion-pion scattering, baryons etc.). For these applications the neglecting by the meson contributions in quark propagator is certainly non-correct from the point of view of the mean-field expansion, and, consequently, the stability of basic model parameters with respect to these contributions becomes a determinative significance.

References

[1] Y. Nambu and G. Jona-Lasinio: Phys.Rev. **122** (1961) 345

[2] T. Eguchi and H. Sugawara: Phys.Rev. D **10** (1974) 4257;
K. Kikkawa: Prog.Theor.Phys. **56** (1976) 947;
H. Kleinert: in "Understanding the Fundamental Constituents of Matter", ed. A. Zichichi, Plenum Press, N.Y., 1978, p.289;
D. Ebert and M.K. Volkov: Z.Phys. C **16** (1983) 305

[3] S.P. Klevansky: Rev.Mod.Phys. **64** (1992) 649

[4] T. Hatsuda and T. Kunihiro: Phys.Reports **247** (1994) 221

[5] E.N. Nikolov et al.: Nucl.Phys. **A608** (1996) 411

[6] E. Quack and S.P. Klevansky: Phys.Rev. C **49** (1994) 3283
D. Ebert, M. Nagy and M.K. Volkov: Yad.Fiz. **59** (1996) 149

[7] M. Oertel, M. Buballa and J. Wambach: Nucl.Phys. **A676** (2000) 247

[8] S. Krewald and K. Nakayama: Annals of Phys. **216** (1992) 201

[9] R.G. Jafarov and V.E. Rochev : Centr.Eur.J.of Phys. **2** (2004) 367 (hep-ph/0311339)

[10] W. Ochs: Plenary talk at Hadron 03 (Aschaffenburg, Germany), hep-ph/0311144

[11] H. Kleinert and B. Van den Bossche: Phys.Lett. **B474** (2000) 336

[12] E. Babaev: Phys.Rev. **D62** (2000) 074020;
G. Ripka: Nucl.Phys. **A683** (2001) 463

[13] V.E. Rochev : J.Phys. A: Math.Gen. **30** (1997) 3671
V.E. Rochev and P.A. Saponov : Int.J.Mod.Phys. **A13** (1998) 3649
V.E. Rochev : J.Phys. A: Math.Gen. **33** (2000) 7379

[14] V.E. Rochev : Talk given at XVII International Workshop on High Energy Physics and Quantum Field Theory (QFTHEP'2003), Samara–Saratov, 4-11 Sept. 2003 (hep-ph/0312004)